Spectral Properties of the SYK Model

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Acknowledgments

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References

A. Garcia-Garcia and J.J.M. Verbaarschot, Spectral and Thermodynamical Properties of the SYK model, Phys. Rev. $\mathbf{D94}$ (2016) 126010 [arxiv:1610.03816].

A. Garcia-Garcia and J.J.M. Verbaarschot, Spectral and Thermodynamical Properties of the SYK model, Phys. Rev. $\mathbf{D96}$ (2017) 066012 [arxiv:1701.06593].

A. Garcia-Garcia, Y. Jia and J.J.M. Verbaarschot, Exact moments of the Sachdev-Ye-Kitaev model up to order $1/N^2$, JHEP (2018) (in press) [arxiv:1801].

A. Garcia-Garcia, Y. Jia and J.J.M. Verbaarschot, Universality and Thouless energy in the supersymmetric Sachdev-Ye-Kitaev Model, Phys. Rev. $\mathbf{D}$ (2018) (in press) [arxiv:1801.01071].

Y. Jia and J.J.M. Verbaarschot, Free Energy and Moments of the Sachdev-Ye-Kitaev Model (2018).

Mario Kieburg, J.J.M. Verbaarschot and Savvas Zafeiropoulos, Dirac Spectra of Two-Dimensional QCD-Like Theories, Phys. Rev. $\mathbf{D90}$ (2014) 085013 [arXiv;1405.0433].

J.J.M. Verbaarschot and M.R. Zirnbauer, Replica Variables, Loop Expansion and Spectral Rigidity of Random Matrix Ensembles, Ann. Phys. $\mathbf{158}$, 78 (1984)

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Introduction

Compound Nucleus

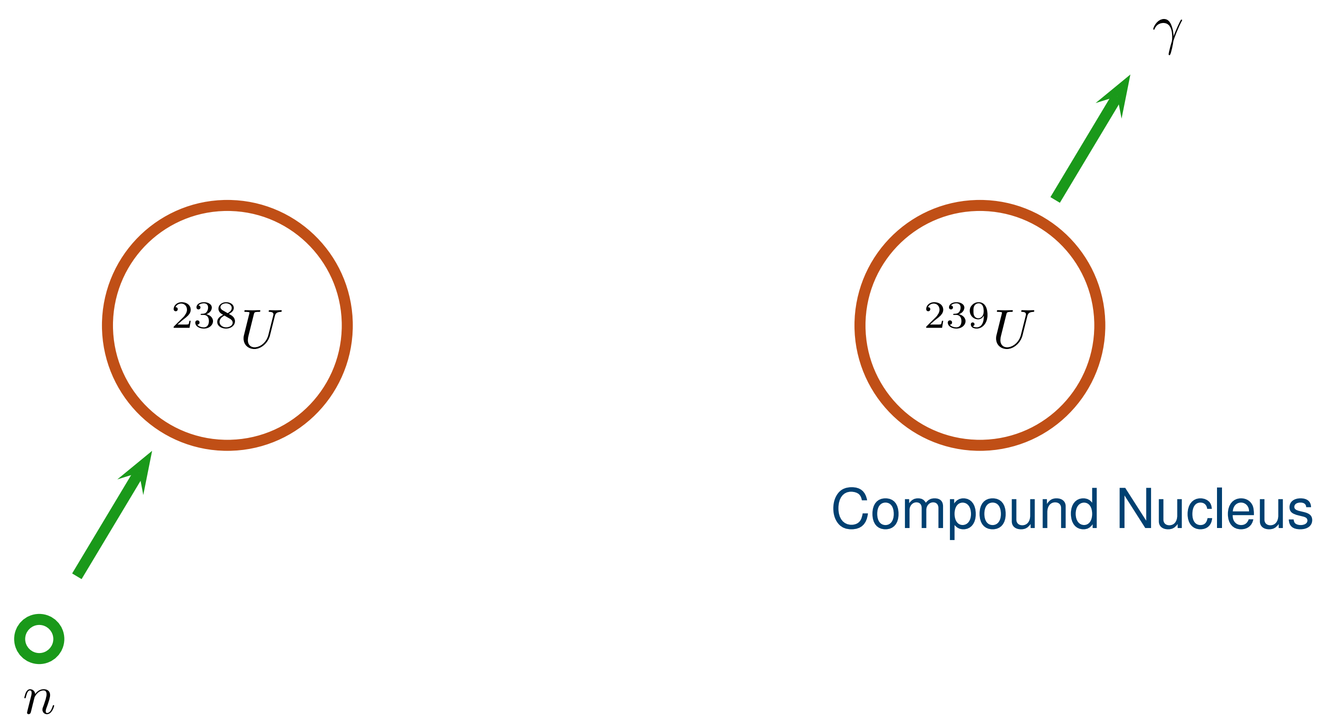
Random Matrix Theory

Two-Body Random Ensemble

Quantum States of Black Hole

- A black hole is a finite system and therefore has discrete quantum states, in fact resonances because they decay.
- All information that goes into a black hole has been scrambled. Therefore, the information content of these quantum states should be minimized.
- What is the density of states?
- What are the correlations of the positions of the resonances?
- Let us have a look at another physical system with similar properties.

Compound Nucleus



- Formation and decay of a compound nucleus are independent.
- Because the system is chaotic, all information on its formation got lost.

Quantum Hair of a Compound Nucleus

Total cross section versus energy (in eV).

Garg-Rainwater-Petersen-Havens,1964

Holography in Nuclear Physics

Because the compound nucleus is chaotic, the fluctuations of the S-matrix are universal so that the average cross-section, $\langle |S_{ab}|^2 \rangle$ only depends on the average diagonal S -matrix elements

$$\langle |S_{ab}|^2 \rangle = \delta_{ab} |\langle S_{ab} \rangle|^2 + \langle |S_{ab}^{\text{fluc}}|^2 \rangle = F_{\text{universal}}(\langle S_{cc} \rangle)$$

with $F_{\text{universal}}$ a universal function. JV-Weidenmüller-Zirnbauer-1983, Mello-Pereyra-Seligman-1984

The average diagonal S -matrix is obtained by an energy average

$$\langle S_{cc}(E) \rangle = \frac{1}{\pi} \int dx \frac{\Gamma}{\Gamma^2 + (E-x)^2} S_{cc}(x),$$

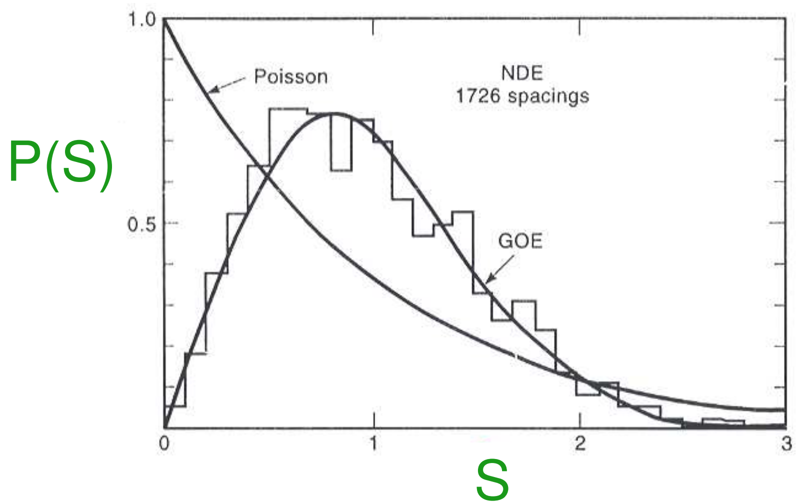
where the width Γ is much larger than the level spacing.

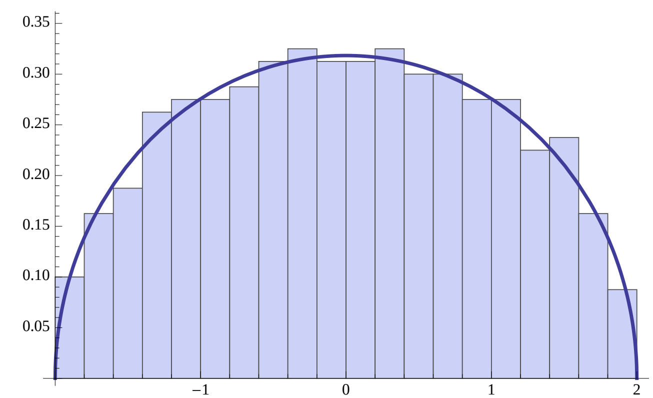
Therefore the average diagonal S matrix is determined by the fast processes, in which a compound nucleus is not formed, i.e. the physics that takes place at the surface of the nucleus.

Compound Nuclei, Chaos and Black Holes

- Most likely a compound nucleus saturates the quantum bound on chaos obtained recently by Maldacena, Shenkar and Stanford. Black holes are believe to saturate this bound as well.
- *Bohigas-Giannoni-Schmidt Conjecture*: If a system is classically chaotic, its eigenvalues are correlated according to random matrix theory.

Nuclear Data Ensemble





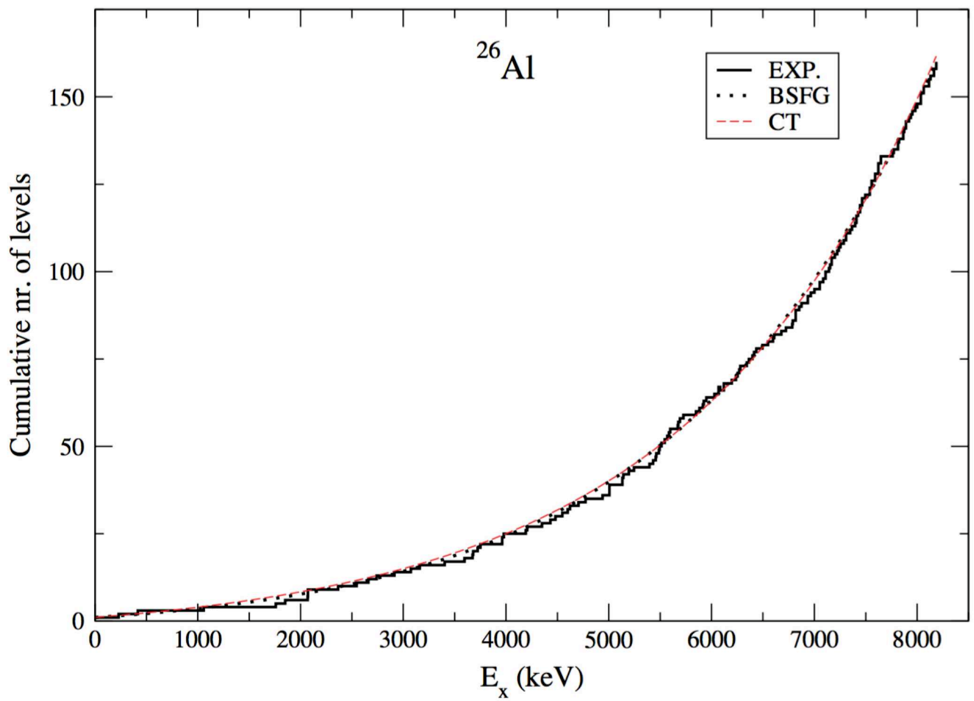
Nearest neighbor spacing distribution of an ensemble of different nuclei normalized to the same average level spacing.

Bohigas-Haq-Pandey, 1983

The eigenvalues of a large class of Random Matrix Ensembles are distributed according to as semi-circle in the limit of very large matrices

Motivation for the Two-Body Random Ensemble

- The nuclear level density behaves as $e^{\alpha\sqrt{E}}$.
- The nuclear interaction is mainly a two-body interaction.
- Random matrix theory describes the level spacings, but it is and N-body interaction with a semicircular level density.



T. von Egidy

Two Body Random Ensemble

$$H = \sum_{\alpha\beta\gamma\delta} W_{\alpha\beta\gamma\delta} a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta.$$

French-Wong-1970

Bohigas-Flores-1971

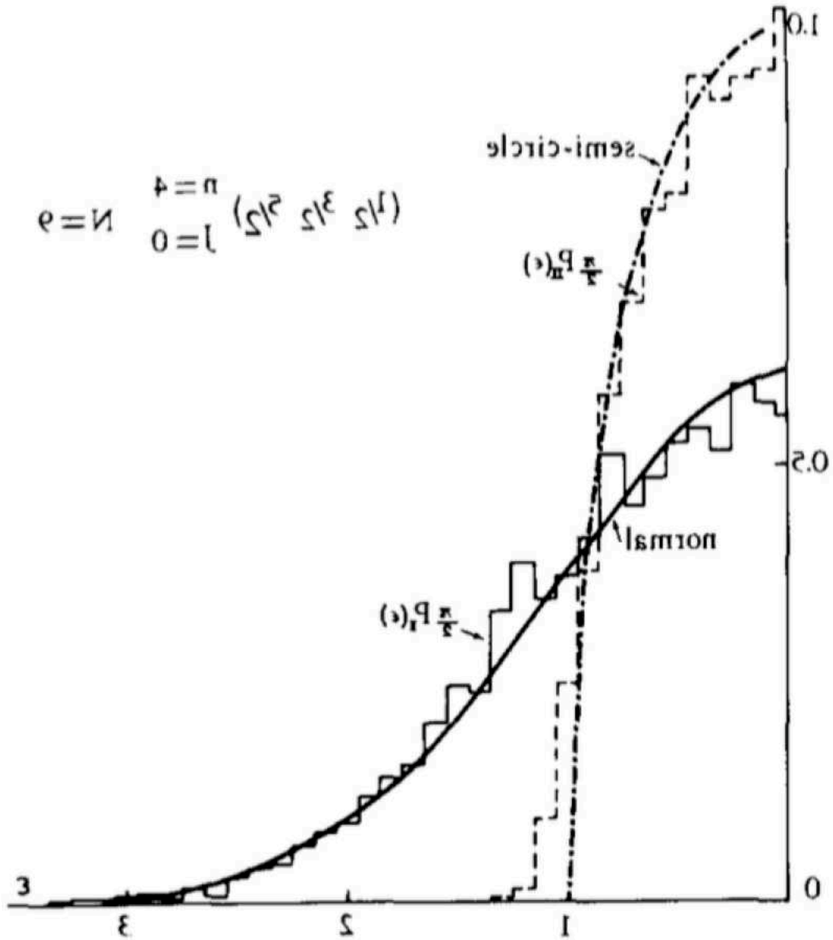
labels of the fermionic creation and annihilation operators run over N single particle states. The Hilbert space is given by all many particle states containing m particles with $m = 0, 1, \cdots, N$.

The dimension of the Hilbert space is: $\sum \binom{N}{m} = 2^N$.

- $W_{\alpha\beta\gamma\delta}$ is Gaussian random.
- The Hamiltonian is particle number conserving.
- The matrix elements of the Hamiltonian are strongly correlated.

Brody-et-al-1981, Brown-Zelevinsky-Horoi-Frazier-1997, Izrailev-1990,Kota-2001,Benet-Weidenmüller-2002,Zelevinsky-Volya-2004, Borgonovi-Izraelev-Santos-Zelevinsky-2016

First Numerical Results



Comparison of the spectral density of the GOE and the two-body random ensemble for the sd-shell. Bohigas-Flores-1971

The Sachdev-Ye-Kitaev Model

The SYK Model

Partition Function

The Sachdev-Ye-Kitaev (SYK) Model

The two-body random ensemble from nuclear physics also has merged into the SYK model, where the fermion creation and annihilation operators are replaced by Majorana operators (in general q of them. For $q = 4$ the model is Sachdev-Ye-1993,Kitaev-2015

$$H = \sum_{\alpha<\beta<\gamma<\delta} W_{\alpha\beta\gamma\delta}\chi_\alpha\chi_\beta\chi_\gamma\chi_\delta, \qquad q = 4.$$

The fermion operators satisfy the commutation relations

$$\{\chi_\alpha, \chi_\beta\} = \frac{1}{2}\delta_{\alpha\beta}.$$

The two-body matrix elements are taken to be Gaussian distributed with variance

$$\sigma^2 = \frac{6}{N^3}.$$

Spectrum and Partition Function

The partition function of N fermions with Hamiltonian H is given by

$$Z(\beta) = \mathrm{Tr} e^{-\beta H} = \int dE \rho(E) e^{-\beta E}.$$

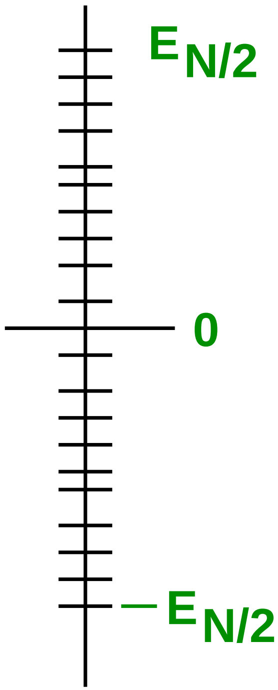
The spectral density is thus given by the Laplace transform of the partition function.

The partition function can be interpreted as the trace of time evolution operator in imaginary time. Feynman told us how to rewrite the time evolution operator as a path integral, Sachdev-Ye-1992

$$Z(\beta) = e^{-\beta F} = \mathrm{Tr} e^{-\beta H} = \int D\chi e^{-\int_0^\beta d\tau \left[\chi \frac{d}{d\tau}\chi + H(\chi)\right]}.$$

where the χ are Grassmann valued functions of τ. This formulation opened a new window on the SYK model. Sachdev-Ye-1992, Kitaev-2015, Maldacena-Stanford-2016, Jevicki-Susuki-Yoon-2016,Bagrets-Altland-Kamenev-2016

Excitations for $q = 2$



Single Particle Energies

- In the ground state, all negative energy states are occupied. So the ground state has $N/2$ Majorana fermions.
- Excitations above the ground state are given by particle hole excitations. So the level density $\sim N$ resulting in a vanishing zero temperature entropy.
- The states of the SYK model for $q \geq 3$ are completely entangled resulting in a level density that is exponentially large in N resulting in a nonvanishing zero temperature entropy. Sachdev-Ye-2009, Sachdev-2010.

Bethe Formula

The level density is given by the Laplace transform of the partition function. The low temperature limit also follows from the Schwarzian action Bagrets-Altland-Kamenev-2016, Stanford-Witten-2017

$$\begin{aligned}\rho(E) &= \int_{r-i\infty}^{r+i\infty} d\beta e^{\beta E} Z(\beta) \\ &= \int_{r-i\infty}^{r+i\infty} d\beta \beta^{-3/2} e^{\beta E} e^{-\beta E_0 + S + \frac{c}{2\beta}}\end{aligned}$$

The integral can be done resulting in

$$\rho(E) = \sinh(\sqrt{2c(E-E_0)}).$$

This gives the Bethe formula for the nuclear level density.

Bethe-1936

Spectral Density of the SYK Model

Large N Limit

Leading Corrections

Analytical Result for the Spectral Density

Bethe Formula

Spectral Density

Spectral density can be calculated in several ways:

- Generating function: Zirnbauer-JV-1984

$$\langle \det(D+z) \rangle.$$

- Inverse Laplace transform of partition function
- Moment method: $\mathrm{Tr} H^{2p}$
  - ⋆ Moments up to order $1/N^3$ have been calculated analytically. Garcia-Garcia-Jia-JV-2018, Jia-JV-2018
  - ⋆ The $1/N^2$ contributions have a simple geometrical interpretation as the total number of triangular loops of an intersection diagram, please ask Yiyang about this.

Spectral Density for even q

The spectral density can be obtained from the moments

$$\langle {\rm Tr} H^{2p} \rangle = {\rm Tr} \langle \left(\sum_\alpha W_\alpha \Gamma_\alpha \right)^{2p} \rangle$$

with Γ_α a product of four gamma matrices. The Gaussian integral is equal to the sum over all pair-wise contractions.

When $2p \ll N$, the Γ_α do not have common gamma matrices and they commute. Since

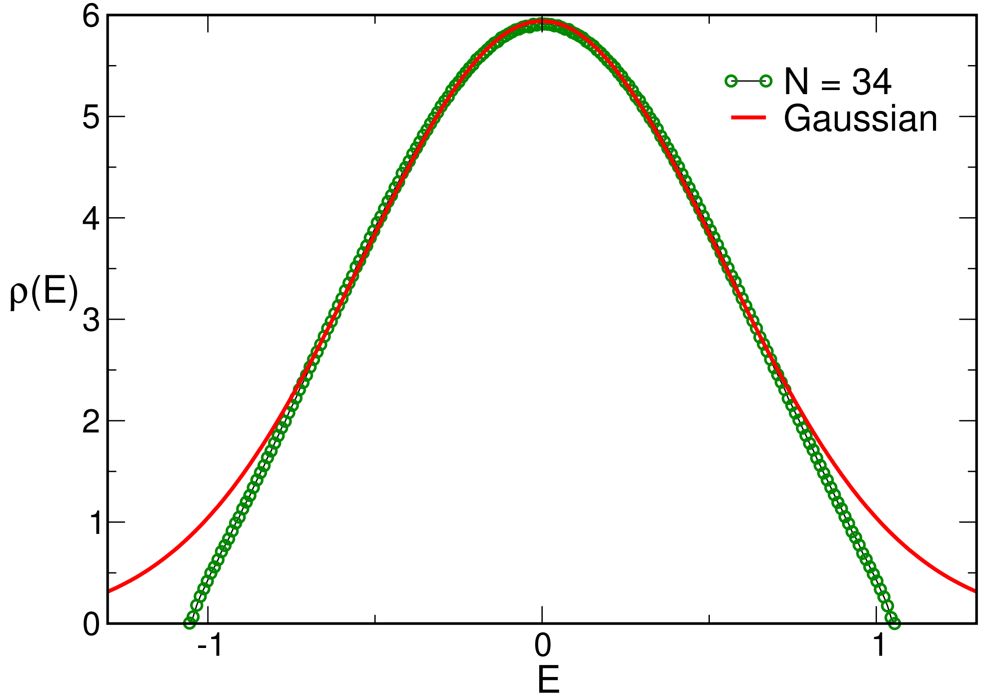
$$\Gamma_\alpha^2 = 1$$

all contractions contribute equally resulting in

$$\langle {\rm Tr} H^{2p} \rangle = (2p-1)!! \langle {\rm Tr} H^2 \rangle)^p$$

which gives a Gaussian distribution. Mon-French-1975, Garcia-JV-2016

Level Density



The center of the spectrum is close to Gaussian but the tail deviates strongly. Garcia-JV-2016

Level Density for odd q

- For odd q the Γ_α anti-commute when they have no gamma matrices in common.
- Contractions are alternating positive and negative while the total number of them is odd so that

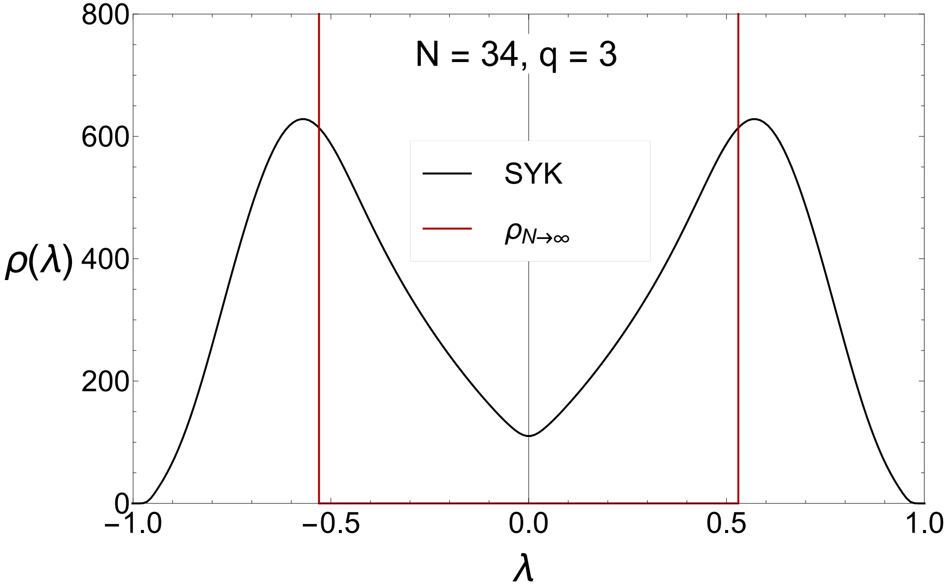
$$\frac{M_{2p}}{M_2^p} = 1$$

These are the moments of two δ functions:

$$\frac{1}{2}(\delta(E-1) + \delta(E+1)).$$

Garcia-Garcia-Jia-JV-2018

Level Density for odd q



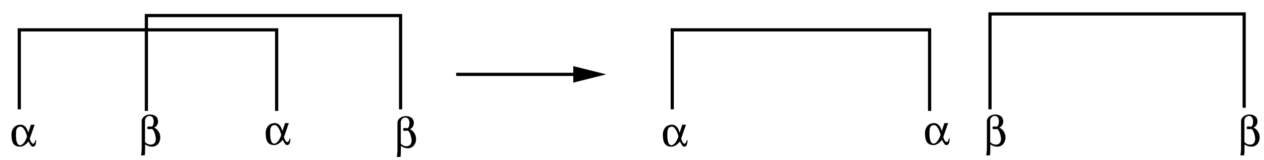
The large N result for the spectral density is not close to the numerical SYK result. Garcia-Garcia-Jia-JV-2018

Moments all orders in q^2/N and $O(1/N^2)$

- For large N, the moments can be calculated exactly if we ignore correlations between contractions.
- A product of four Majorana operators satisfies the commutation relations Garcia-Garcia-JV-2016

$$\Gamma_\alpha \Gamma_\beta + (-1)^p \Gamma_\beta \Gamma_\alpha = 0,$$

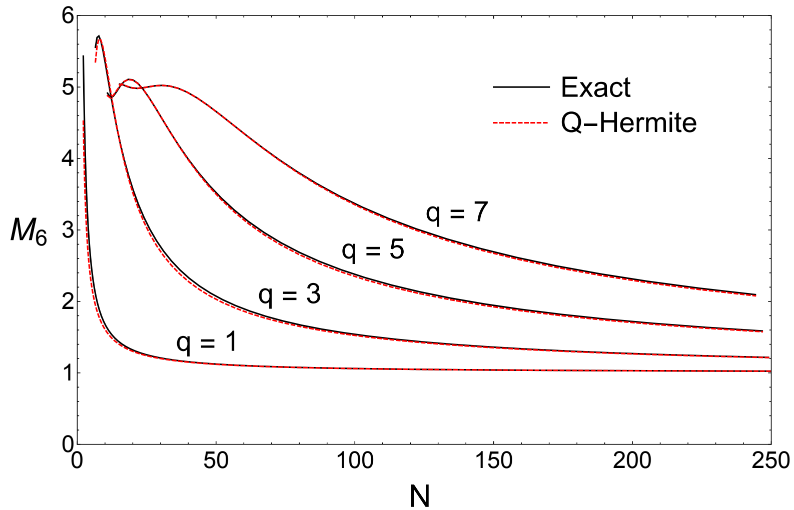
where p is the number of γ-matrices they have in common.

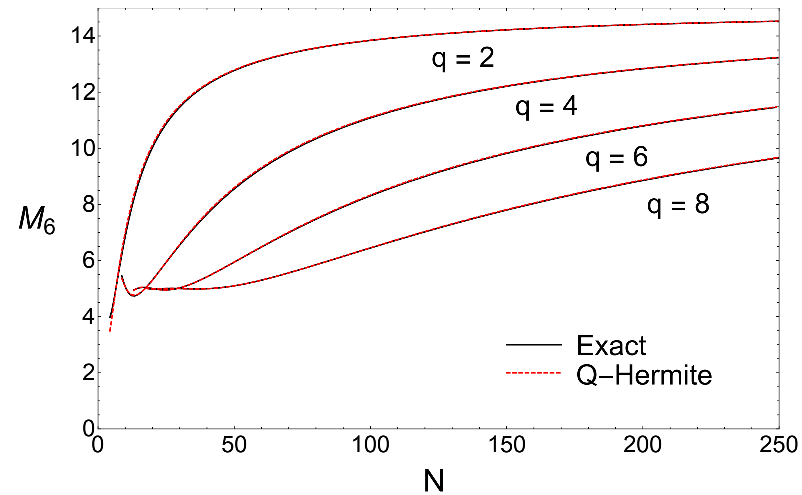


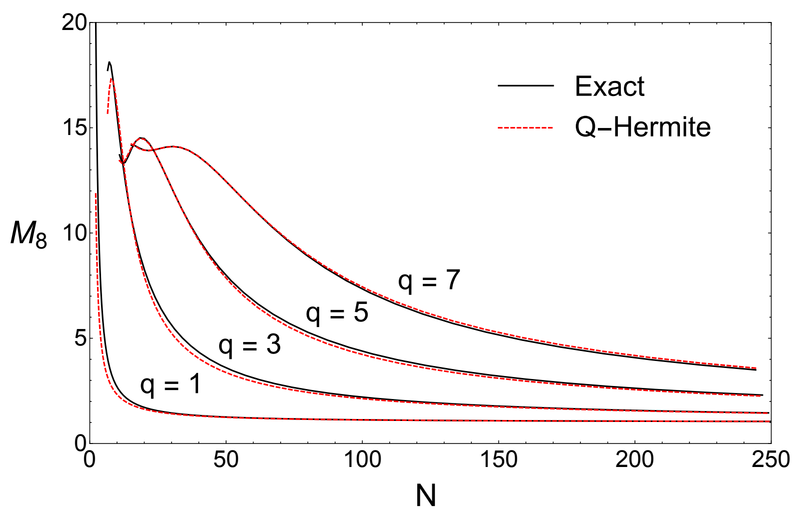
This results in the suppression factor of intersecting relative to nested contractions

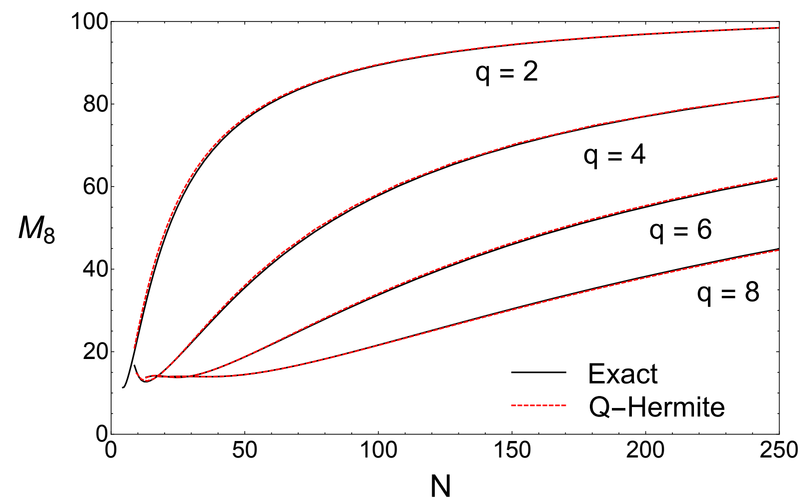
$$\eta_{N,q} = \binom{N}{q}^{-1} \sum_p (-1)^p \binom{q}{p} \binom{N-q}{q-p} \sim (-1)^q e^{-2q^2/N}..$$

N Dependence of Sixth and Eight Moments









Spectral Density at Finite N

If α is the number of intersections, the moments are given by

$$\frac{M_{2p}}{M_2^p} = \sum_{\text{contractions}} \eta^\alpha = \frac{1}{(1-\eta)^p} \sum_{k=-p}^{p} (-1)^k \eta^{k(k-1)/2} \binom{2p}{p+k},$$

where the sum has been evaluated by the Riordan-Touchard formula.

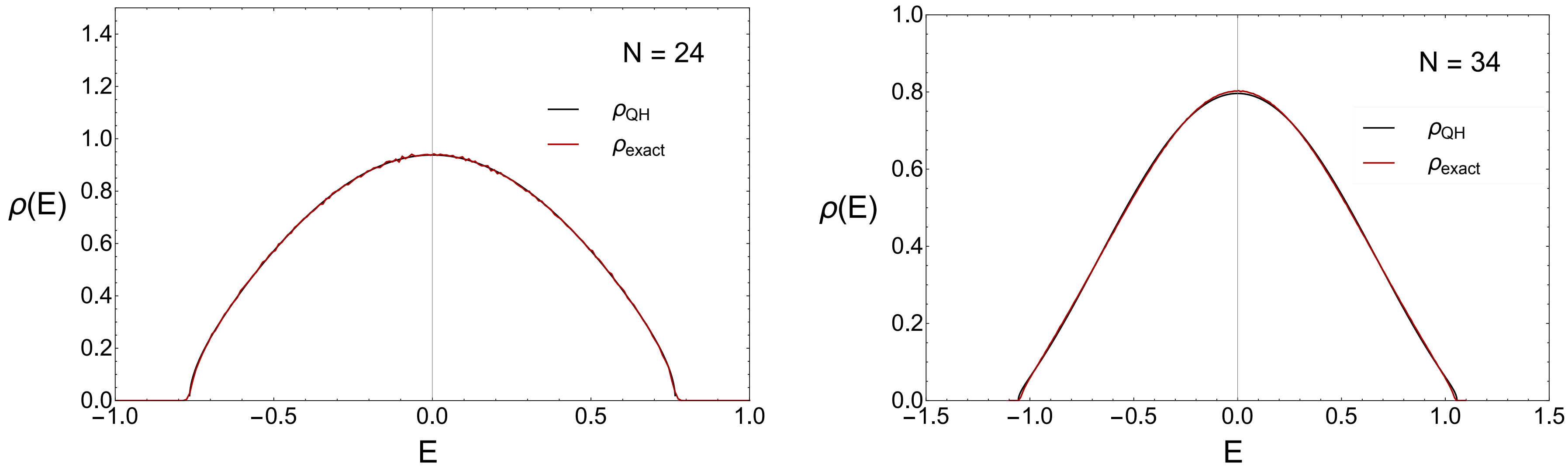
Erdos-2014, Cotler-et-al-2016, Garcia-Garcia-JV-2017

These are the moments corresponding to the weight function of the Q-Hermite Polynomials. This results in the spectral density

$$\rho_{\mathrm{QH}}(E) = c_N \sqrt{1-(E/E_0)^2} \prod_{k=1}^{\infty} \left[1 - 4\frac{E^2}{E_0^2}\left(\frac{1}{2+\eta^k+\eta^{-k}}\right)\right]$$

with $E_0^2 = \frac{4\sigma^2}{1-\eta}$ and σ the variance of the spectral density. This includes $1/N$ corrections and all orders in q^2/N .

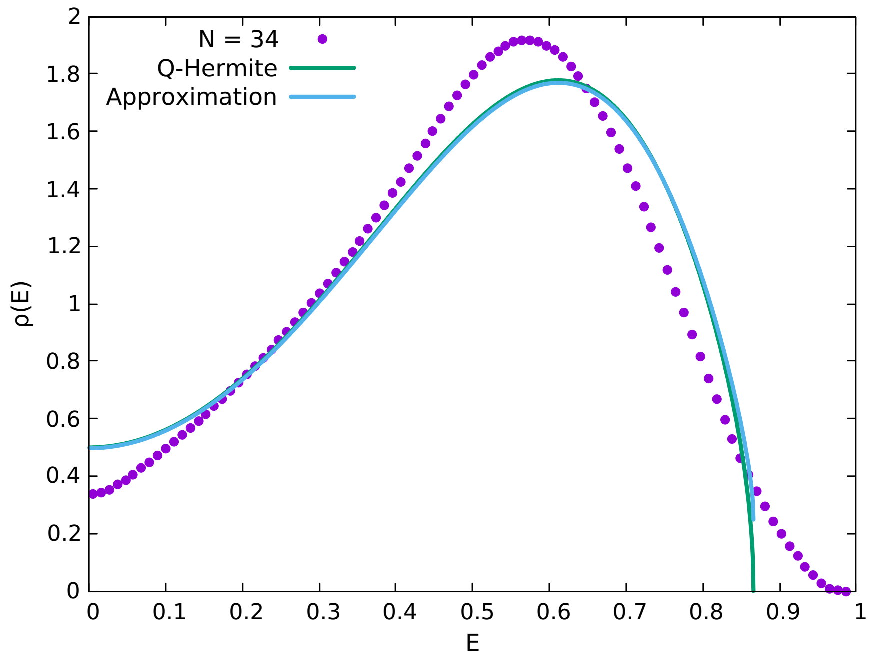
Comparison with Numerical Results for $q = 4$

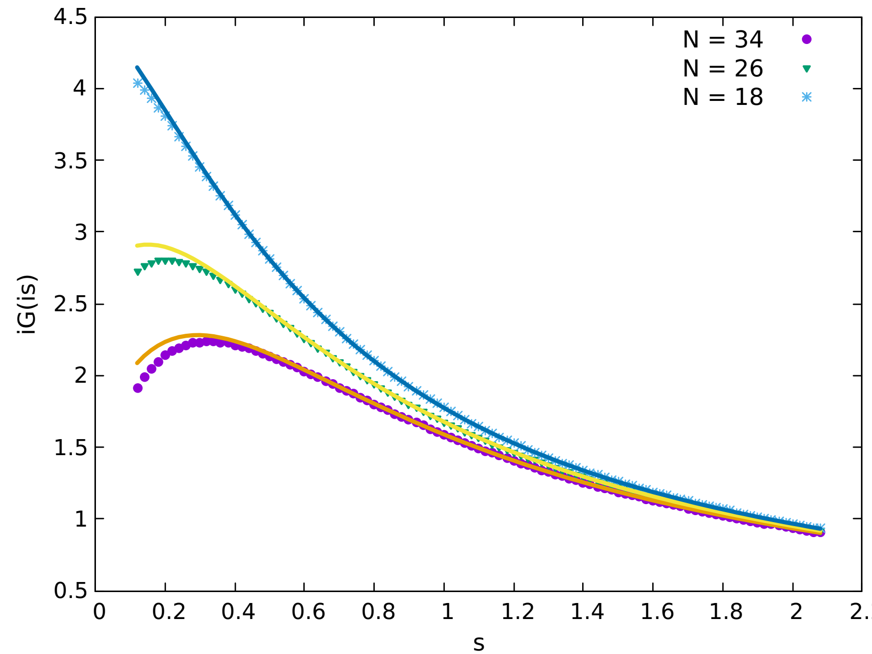


Comparison of the exact spectral density obtained by numerical diagonalization and the Q-Hermite result for the spectral density.

Garcia-Garcia-JV-2017

Comparison with Numerical Results for $q = 3$





Comparison of the exact spectral density (left) and the resolvent (right) obtained by numerical diagonalization compared to the Q-Hermite result.

Garcia-Garcia-JV-2017

Large N Approximation for even q

For $N \gg 1$, E not close to $\pm E_0$, this simplifies to Garcia-Garcia-JV-2017

$$\rho_{\rm asym}^{q\ \textsf{even}}(E) \quad = \quad c_N \exp\left[\frac{2\arcsin^2(E/E_0)}{\log\eta}\right].$$

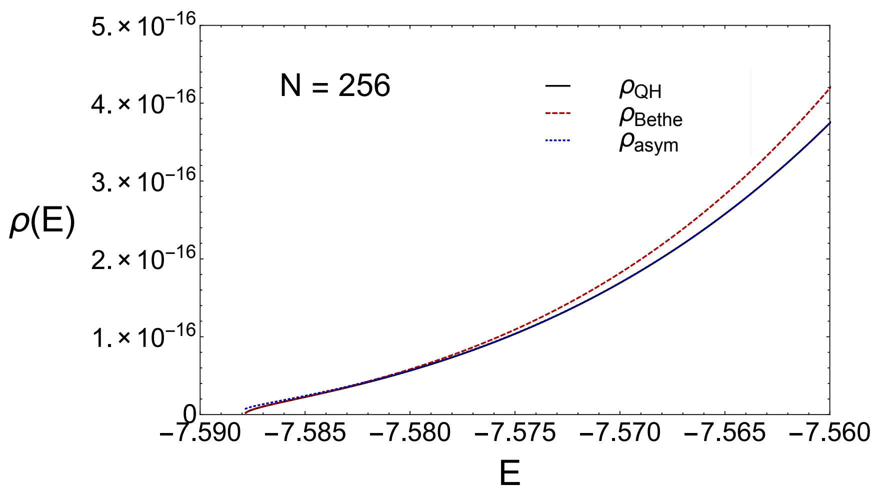
Very close to the ground state the level density is given by so that Cotler-et-al-2016, Garcia-JV-2017,Altland-Bagrets-Kamenev-2017

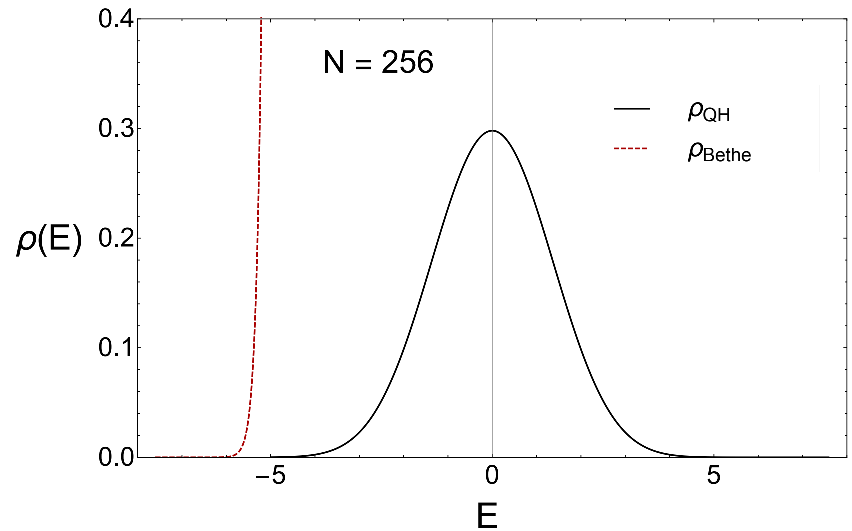
$$\rho(E) = e^{\frac{N}{2}\log 2 - \frac{N}{q^2}\frac{\pi^2}{4}} \sinh\left[\frac{\pi N}{2q^2}\sqrt{2(E-E_0)/|E_0|)}\right].$$

For odd q the large N limit of the spectral density away from the edge of the spectrum is given by

$$\rho_{\rm asym}^{q\ \textsf{odd}}(E) = e^{N/2\log 2 - N\pi^2/16q^2} \cosh(\frac{\pi\arcsin E/E0}{\log|\eta|} \exp\left[2\frac{\arcsin^2 E/E_0}{\log|\eta|}\right].$$

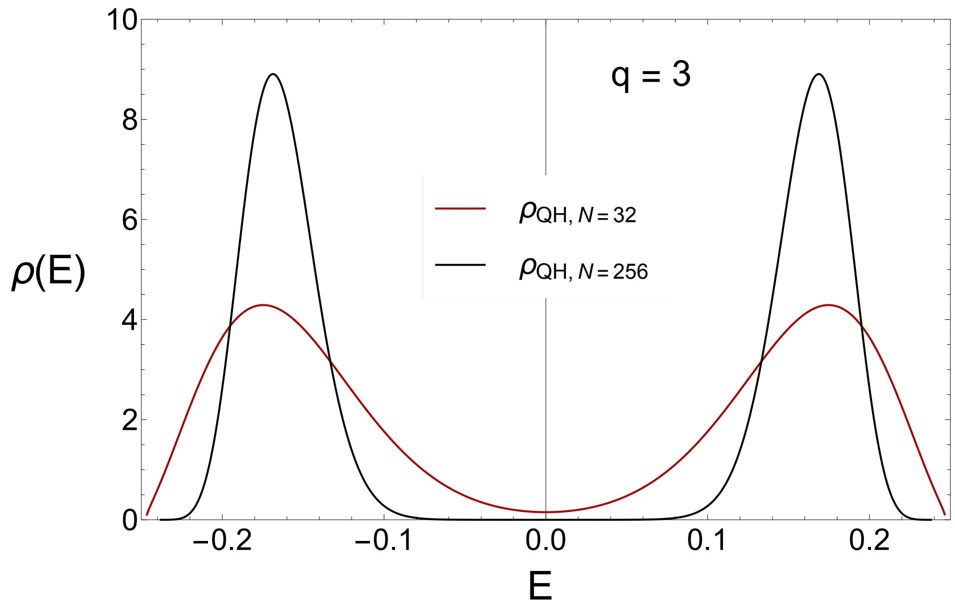
Comparison to the Bethe Formula for $q = 4$





Comparison of the exact Q-Hermite result to the Bethe Formula, $\rho(E) \sim \sinh\sqrt{2c(E - E_0)}$. The Bethe formula is valid in the very tail where the density is non-Gaussian. Note that the number of states in the tail $\exp[N/2\log 2 - N\pi^2/4q^2]$ is exponentially large but also exponentially suppressed with respect to the states in the bulk.

N -Dependence of the Spectral Density for $q = 3$



For large N the spectral density for odd q approaches the sum of two delta-functions Garcia-Garcia-Jia-JV-2018. The spectral density around zero increases exponentially with N, but is also exponentially smaller than the density in the bulk.

Level Density and Free Energy

$$Z(\beta) = 2^{N/2} \int e^{-\beta E} \exp\left[\frac{2\arcsin^2(E/E_0)}{\log\eta}\right].$$

For large N this integral can be evaluated by a saddle point approximation. The result is given by

$$\begin{aligned}\beta F &= \frac{2}{\log\eta}\pi v \tan\frac{\pi v}{2} - \frac{(\pi v)^2}{2\log\eta} \quad \text{with} \quad \beta\mathcal{J} = \frac{\pi v}{\cos\frac{\pi v}{2}}, \\ &\sim -\beta E_0 - \frac{N}{2}\log 2 + \frac{N}{q^2}\frac{\pi^2}{4} - \frac{1}{\beta}\frac{N}{q^2}\frac{\pi^2}{q^2 E_0/N} + O(1/\beta^2)\end{aligned}$$

and $v = (2/\pi)\arcsin \bar{E}/E_0$ and $\mathcal{J} = (E_0/2)\log\eta$. This is exactly the large N large q result obtained by Maldacena and Stanford valid for all finite β . Maldacena-Stanford-2015.

The $1/N^2$ corrections to the moments give the $1/q$ corrections to the free energy obtained by Tarnopolsky (2018) Jia-JV-2018, Garcia-Garcia-Jia-JV-2018

Free Energy for odd q

$$Z(\beta) = \int dE \rho^{\mathsf{odd}q}_{\mathsf{asym}}(E) e^{-\beta E^2}.$$

For large N this integral can be evaluated by a saddle point approximation resulting in the free energy

$$\beta F = \frac{N}{2}\log 2 + \frac{v^2}{8\log|\eta|} + \frac{v}{2\log|\eta|}\tan((\pi - v)/4).$$

where v satisfies the saddle point equation

$$\frac{v}{\mathcal{J}\cos(v/2)} - \beta = 0.$$

This agrees with with the path integral result for all finite β and to order $1/q^2$ obtained in Fu-Gaiotto-Maldacena-Sachdev-2017.

The $1/q$ corrections of the high temperature limit of this result corrections follow from the $1/N^2$ corrections of the moments.

Spectral Correlations of the SYK Model

Spectral Correlators

Spectral Correlations for $q = 4$

Spectral Correlations for $q = 3$

Spectral Correlations

It has been shown hat the SYK model in maximally chaotic, in the sense that the Lyapunov exponent saturates the bound of

$$\lambda_L \leq \frac{2\pi kT}{\hbar}.$$

Kitaev-2015, Maldacena-Shenker-Stanford-2016

- If this is the case its spectrum should behaves as a quantum chaotic system, i.e. the eigenvalue correlations are given by random matrix theory with the corresponding random matrix ensemble determined by the anti-unitary symmetries.
- The anti-unitary symmetries follow from the anti-unitary symmetries of the gamma matrices which carry over the Bott periodicity to the SYK Hamiltonian. You-Ludwig-Xu-2016, Garcia-Garcia-JV-2016,Fu-Gaiotto-Maldacena-Sachdev-2017, Li-Liu-Xin-Zhou-2017, Kanazawa-Wettig-2017

Unfolding

- The average spectral density is not universal, and for comparison with random matrix theory, the dependence of the spectral fluctuations on the average spectral density have to be eliminated. This is essential.
- This is achieved by unfolding the spectrum, i.e. by mapping the spectrum by a smooth transformation to one with spectral density equal to 1.
- To better see the correlations one also subtracts to disconnected part of the correlation functions.

Spectral Observables

- $P(S)$: the distribution of the spacing of consecutive levels.
- $\Sigma^2(L)$: the variance of the number eigenvalues in an interval that contains L levels on average.
- To increase statistics we can perform a spectral average in addition to the ensemble average for the calculation of of $P(S)$ and $\Sigma^2(L)$.
- Spectral form factor

$$g(\beta, t) = \frac{1}{Z^2(\beta)} \sum_{k,l} e^{-(\beta+it)E_k-(\beta-it)E_l} \approx \frac{Z(2\beta)}{Z^2(\beta)}$$

(For long times the sum is dominated by the diagonal terms with $k = l$)

Spectral Correlations

Spectral Density

$$\rho(x) = \left\langle \sum_k \delta(x - E_k) \right\rangle .$$

Two point correlation function

$$\begin{aligned}
\rho_2(x, y) &= \left\langle \sum_{kl} \delta(x - E_k)\delta(x - E_l) \right\rangle \\
&= \delta(x - y)\rho(x) + \left\langle \sum_{k \neq l} \delta(x - E_k)\delta(x - E_l) \right\rangle .
\end{aligned}$$

The first term is due to self-correlations.

The connected correlator is given by

$$\rho_{2c} = \rho_2(x, y) - \rho(x)\rho(y).$$

Number Variance and Spectral Form Factor

Number variance

$$\Sigma^2(n) = \int_{x_0}^{x_0+n} \int_{x_0}^{x_0+n} dx dy \rho_{2c}(x,y)$$

$$\Sigma^2_{\rm self}(n) = n.$$

Unfolded spectral form factor

$$g(\beta,t) = \beta^2 \int dx dy e^{-(\beta+it)x-(\beta-it)y} \rho_2(x,y).$$

Can be split into a connected part, a disconnected part and a part due to the self correlations. The part due to self correlations is given by

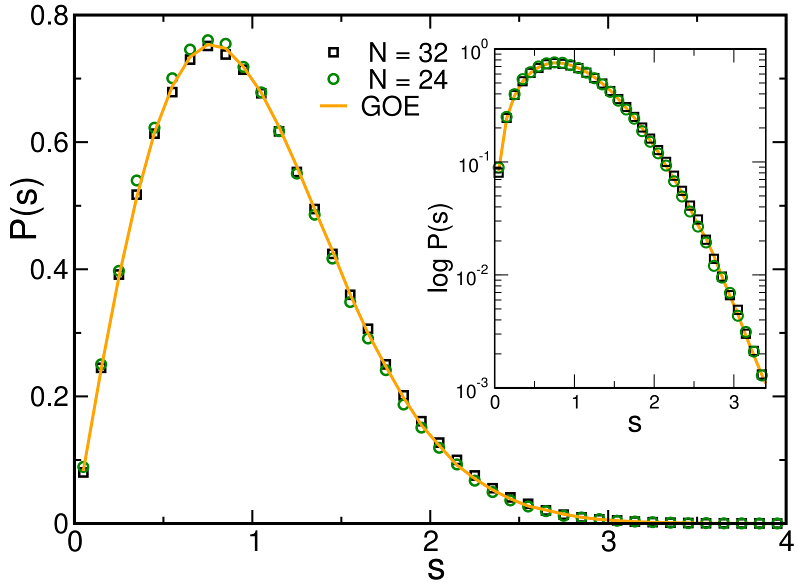
$$g_{\rm self}(t) = \int dx \rho(x) e^{-2\beta x} = \text{constant}.$$

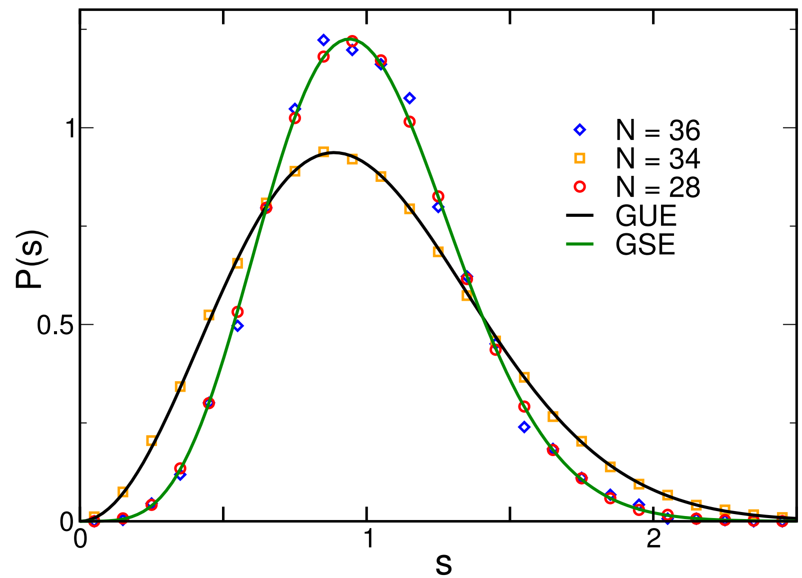
Examples of Spectral Form Factor and Number Variance

$\rho_2(x,y)$ Two-point function	$g(t)$ Form factor	$\Sigma^2(L)$ Number variance
$\rho_2^{\mathsf{GUE}} \approx \delta(x-y) - \frac{1}{2\pi^2(x-y)^2}$	$g(t) = t\theta(1-\|t\|) + \theta(1-\|t\|)$	$\Sigma^2(n) \sim \frac{1}{\pi^2}\log n$
$\rho_2(x,y) = \delta(x-y)$	$g(t) = \frac{\beta}{2}$	$\Sigma^2(n) = n$
$\rho_c(x,y) = c$	$g(t) = c\frac{\beta^2}{\beta^2+t^2}$	$\Sigma^2(n) = cn^2$

Spectral Correlators for $q = 4$

Nearest Neighbor Spacing Distribution



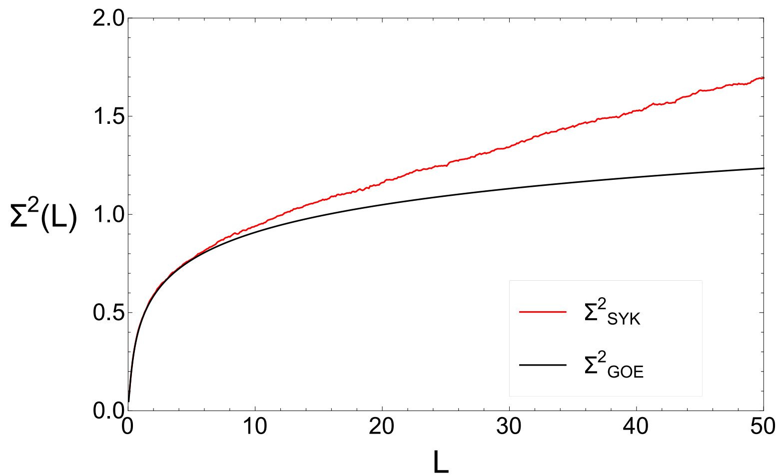


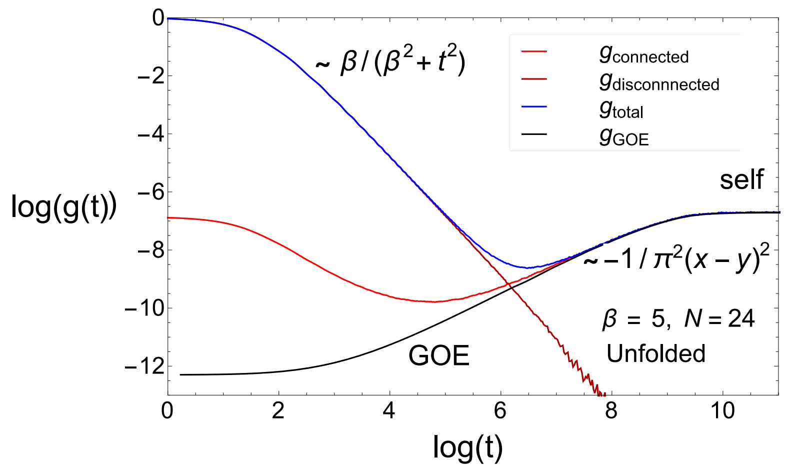
Nearest neighbor spacing distribution for the bottom (left) and bulk part of the spectrum compared to random matrix theory.

Garcia-Garcia-JV-2016, Garcia-Garcia-JV-2017

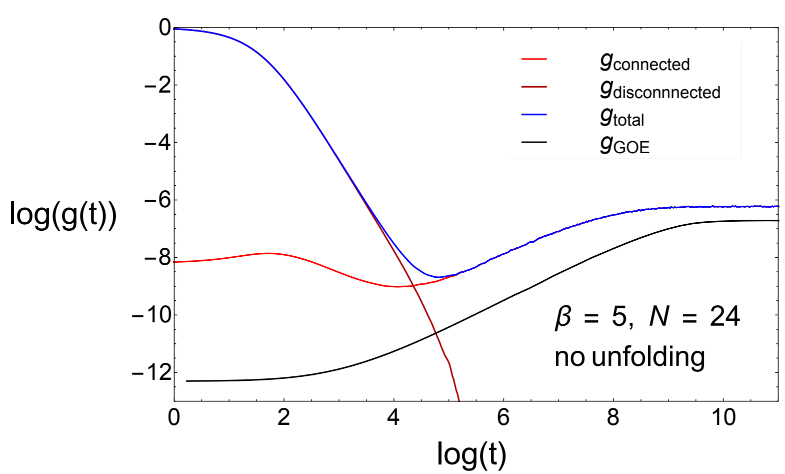
This is in agreement with results for the distribution of the ratio of consecutive spacings. You-Ludwig-Xu-2016

Number Variance Versus Spectral Form Factor





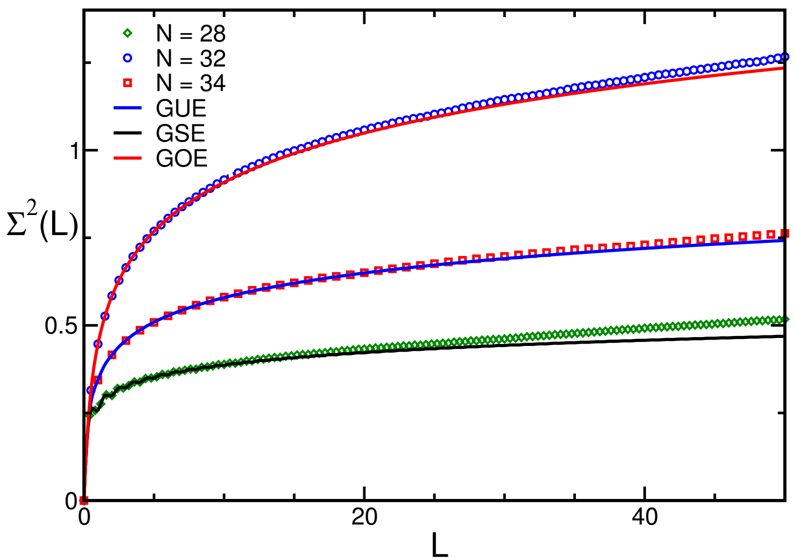
Number variance (left) and spectral form factor (right). $\Sigma^2(L)$ is calculated starting at the 50th eigenvalue above the ground state.

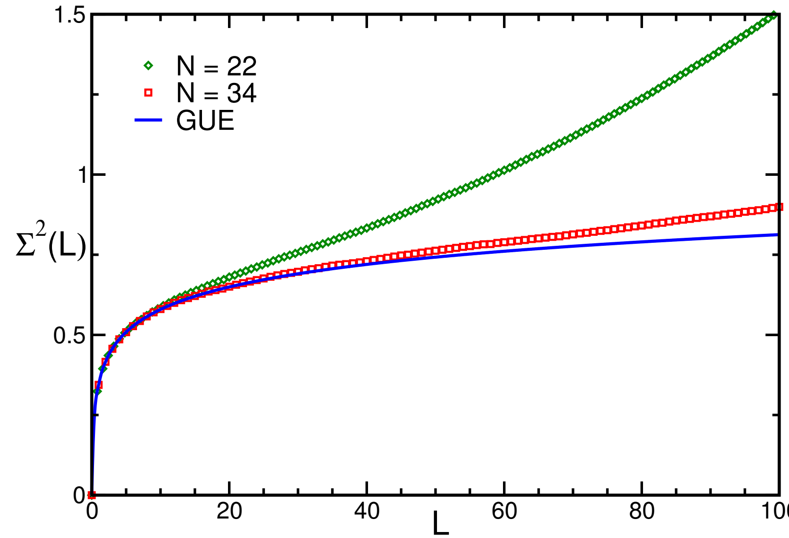


Garcia-Garcia-JV-arXiv:1610.02363

Cotler-et-al-arXiv:1611.04650

Number Variance in the Bulk



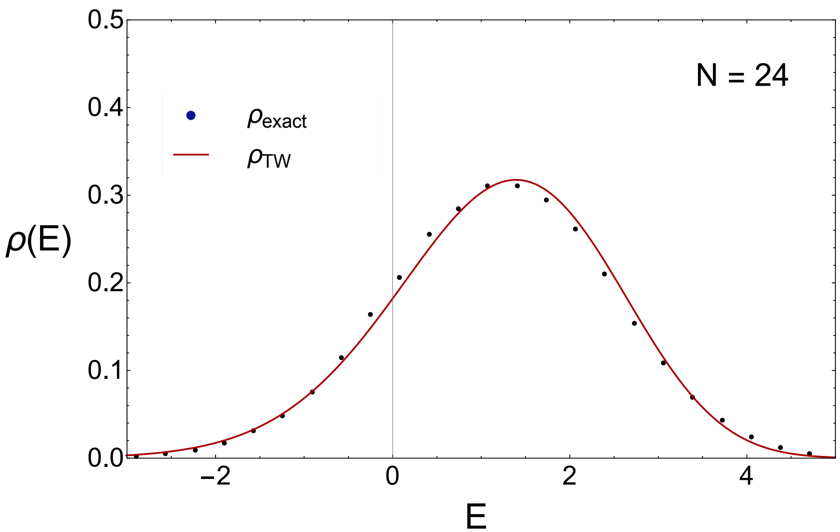


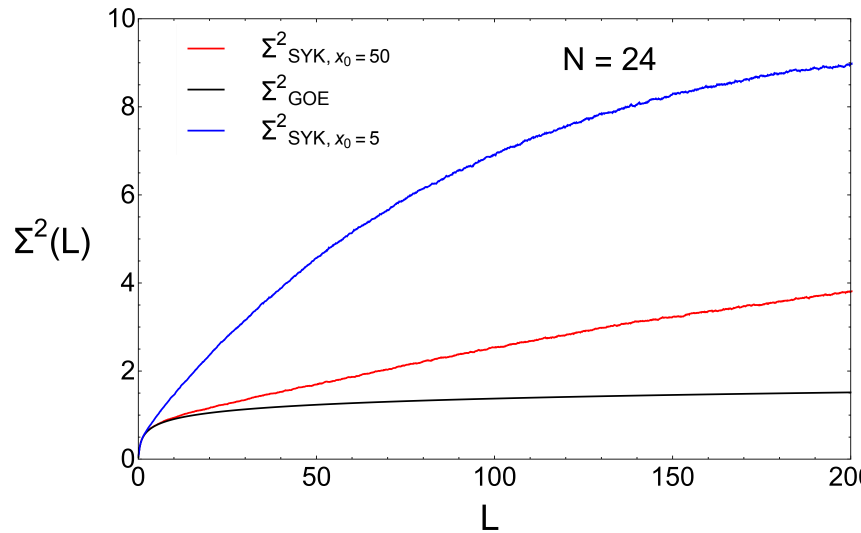
Garcia-Garcia-JV–arXiv:1610.02363

These results have been confirmed by an independent collaboration who calculated the spectral form factor which is the Fourier transform of the spectral correlator.

Cotler-Gur-Ari-Hanada-Polchinski-Saad-Shenker-Streicher-Tezuka-arXiv:1611.04650

Tracy-Widom Distribution



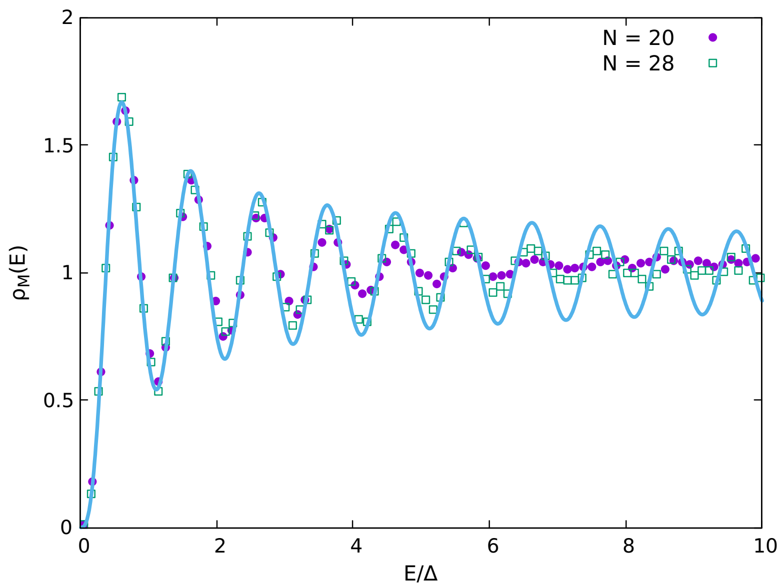


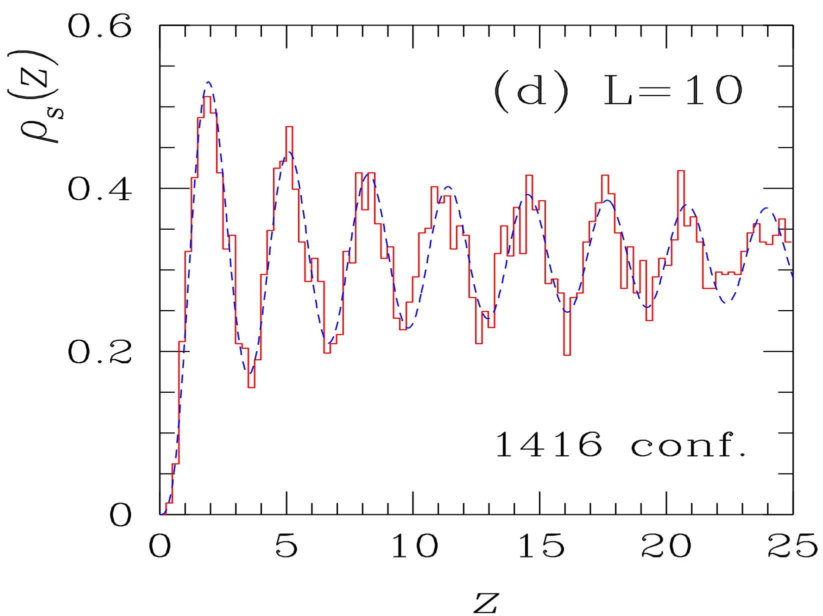
Distribution of the ground state energy compared to the Tracy-Widom distribution of the Gaussian Orthogonal Ensemble. There is no fitting – the parameter of the Tracy-Widom distribution is fixed by equating its expectation value to the numerical one, at the point $E = 0$, is edge of the spectrum as predicted by the Q-Hermite expression.

Garcia-Garcia-JV-2017

Spectral Correlators for $q = 3$

Microscopic Spectral Density for odd q

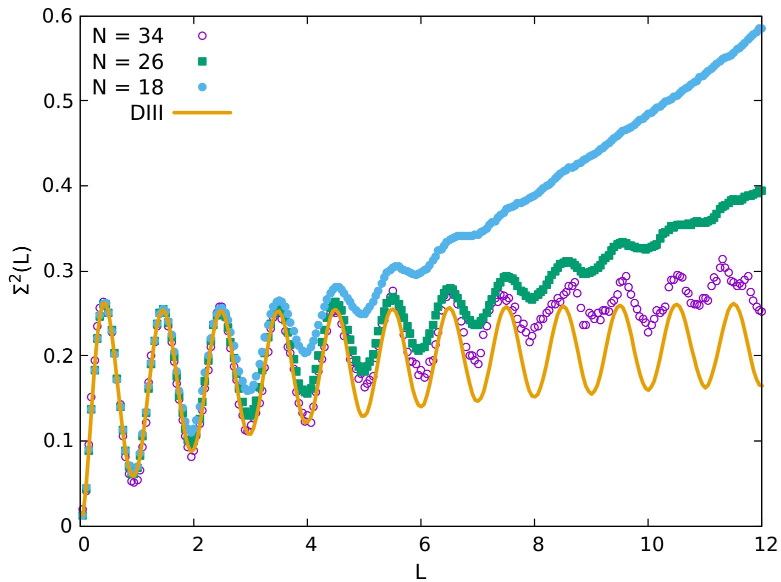


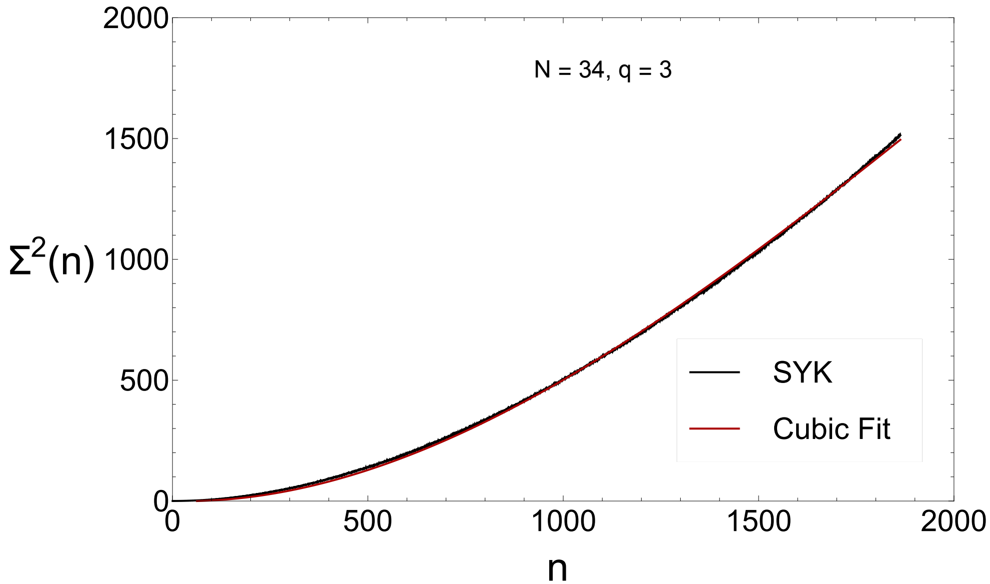


The spectral density on the scale of the average level spacing, SYK left (Garcia-Garcia-Jia-JV-2018) and two color lattice QCD right (Wettig-JV-et-al-1996).

- Because of the choral symmetry the microscopic spectral density shows universal oscillations – of the same kind as in lattice QCD.
- The Thouless energy scales linear in N .

Spectral Correlations of the Supercharge for odd q

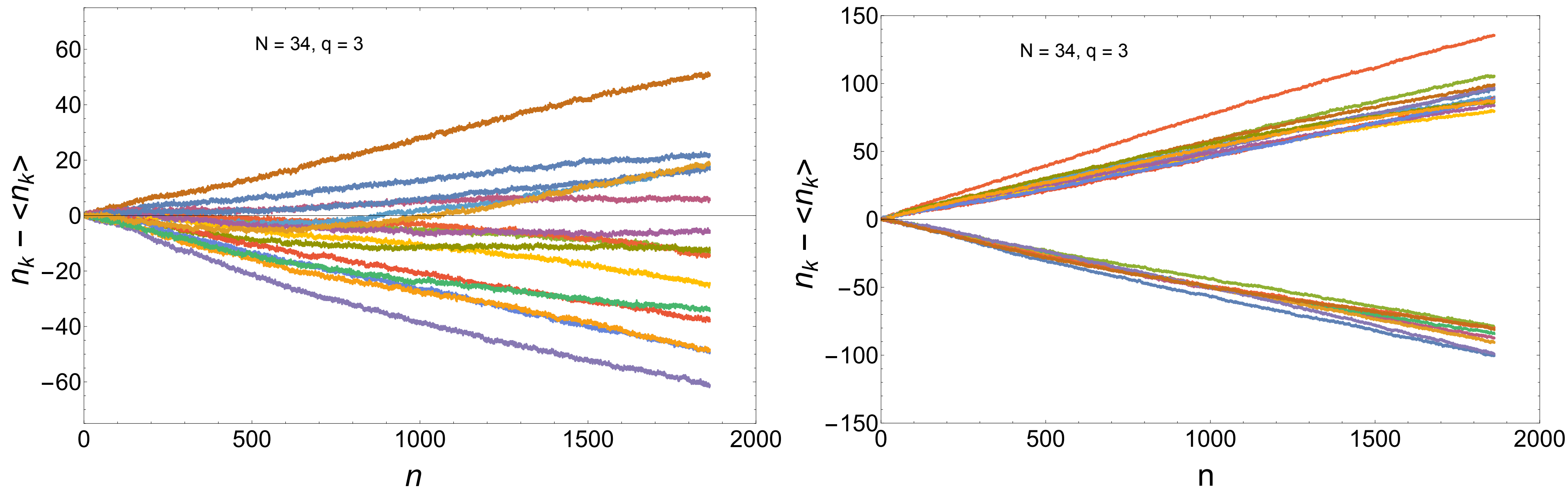




- The Thouless energy is proportional to N. Disagrees with the N^2 dependence found for even q. Garcia-Garcia-JV-2016, Bagrets-Altland-2017

- The number variance is mainly a quadratic function of L with a small linear and cubic correction

Meaning of the Quadratic Term



- A linear deviation from the average number of levels gives a quadratic number variance.
- The quadratic term is entirely due to ensemble fluctuations. We would have found RMT behavior to very large distances if we would have unfolded the spectrum for each configuration separately. However, this is not the correct procedure.

Implications

- The SYK model for $q = 3$ does not possess spectral ergoditicy, and ensemble averaged observables may be different from the observable for an individual matrix.
- Spectra for $q = 3$ are very soft. This might indicate that the replica trick does not work.